

ESTIMATES OF INVARIANT METRICS ON PSEUDOCONVEX DOMAINS NEAR BOUNDARIES WITH CONSTANT LEVI RANKS

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ABSTRACT. Estimates of the Bergman kernel and the Bergman and Kobayashi metrics on pseudoconvex domains near boundaries with constant Levi ranks are given.

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1. INTRODUCTION

In this paper we study boundary behavior of the Bergman kernel and invariant metrics on pseudoconvex domains near boundaries whose Levi-forms have constant ranks. Our main theorem can be stated as follows:

Theorem 1.1. *Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain and let $z^0 \in b\Omega$. Let $\delta(z)$ be the Euclidean distance to the boundary. Let $K_\Omega(z, z)$ be the Bergman kernel and let $F_\Omega(z, X)$ be either the Bergman or the Kobayashi metric. Assume that the boundary $b\Omega$ is smooth in a neighborhood V of z^0 and its Levi-form has constant rank $n - l - 1$, $0 \leq l \leq n - 1$, for all $z \in V \cap b\Omega$. Then there exist a neighborhood $W \subset \subset V$ of z^0 and positive constants C_1, C_2 , and C_3 such that*

$$C_1^{-1}|\delta(z)|^{-(n-l+1)} \leq K_\Omega(z, z) \leq C_1|\delta(z)|^{-(n-l+1)}$$

and

$$C_2^{-1}(M(z, X) + C_3|X|^2) \leq (F_\Omega(z, X))^2 \leq C_2(M(z, X) + C_3|X|^2)$$

for all $z \in W \cap \Omega$ and $X \in T_z^{1,0}(\Omega)$, where

$$M(z, X) = \frac{|L_\delta(z, X)|}{|\delta(z)|} + \frac{|\langle \partial\delta(z), X \rangle|^2}{|\delta(z)|^2}.$$

Limiting asymptotic behavior on smoothly bounded, strongly pseudoconvex domains in \mathbb{C}^n was obtained by Diederich [D] for the Bergman kernel and metric, and by Graham [G] for the Carathéodory and Kobayashi metrics. In a celebrated paper [Fe], Fefferman obtained asymptotic formulas for the Bergman kernel and metric and used them to establish smooth extension of biholomorphic maps. Fefferman type asymptotic expansions for the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains were given in [F2] (see also [Ma] for related results). Estimates for the Bergman kernel and the invariant metrics, in terms of big constants and small constants, were obtained by Catlin [C2] for smooth bounded pseudoconvex domains of finite type in \mathbb{C}^2 and by J.-H. Chen [Ch] and McNeal [M] for convex domains of finite type in \mathbb{C}^n . We refer the reader to the monograph of Jarnicki and Pflug [JP] for extensive treatment on the subject.

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Strong pseudoconvexity and Levi-flatness are in a sense at the opposite ends of pseudoconvexity. Theorem 1.1 shows that in terms of boundary behavior of the invariant metrics, these two types of domains bear striking resemblance. Our proof of Theorem 1.1 uses an idea of Catlin (see [C2]): We construct plurisubharmonic functions whose complex Hessians blow up at the rate of $1/\delta^2(z)$ in the complex normal direction and at a rate of $|L_\delta(z, X)|/\delta(z)$ in the complex tangential directions. To estimate the Kobayashi metric from below, we also use the Sibony metric [S].

This paper is organized as follows. In Section 2, we recall the necessary definitions and basic properties of the Bergman kernel and the invariant metrics. In Section 3, we review the local foliation of a hypersurface with constant Levi rank and choose local holomorphic coordinates under which the defining function has a desirable form. Plurisubharmonic functions with large Hessians are constructed in Section 4. Theorem 1.1 is proved in Section 5.

Throughout the paper, we will use C to denote positive constants which may be different in different appearances. We will also use $f \gtrsim g$ to denote $f \geq Cg$ where C is a constant independent of relevant parameters, and use $A \approx B$ to denote $A \gtrsim B$ and $B \gtrsim A$.

2. PRELIMINARIES

Let Ω be a domain in \mathbb{C}^n and let \mathbb{D} be the unit disc. Let $H(\mathbb{D}, \Omega)$ be the set of holomorphic maps from \mathbb{D} into Ω . Let $z \in \Omega$ and let $X = \sum_{i=1}^n X_i \partial/\partial z_i \in T_z^{1,0}(\Omega)$. (We sometimes identify $T_z^{1,0}(\Omega)$ with \mathbb{C}^n without explicit notices.) The Kobayashi metric is given by

$$F_\Omega^K(z, X) = \inf \left\{ 1/\lambda; \quad f \in H(\mathbb{D}, \Omega), f(0) = z, f'(0) = \lambda X, \lambda > 0 \right\}.$$

Let $A^2(\Omega)$ be the space of square-integrable holomorphic functions on Ω . The Bergman kernel (on the diagonal) and metric can be defined via the following extremal properties:

$$K(z, z) = \sup \left\{ |f(z)|^2; \quad f \in A(\Omega), \|f\|_{L^2} \leq 1 \right\}$$

and

$$F_\Omega^B(z, X) = \frac{\sup \left\{ |Xf(z)|; \quad f \in A(\Omega), f(z) = 0, \|f\|_{L^2} \leq 1 \right\}}{K(z, z)^{1/2}}.$$

Denote by $\mathcal{S}_z(\Omega)$ the class of functions u defined on Ω such that: (1) u is C^2 on a neighborhood of $z \in \Omega$; (2) $0 \leq u \leq 1$ on Ω and $u(z) = 0$; and (3) $\log u$ is plurisubharmonic on Ω . The Sibony metric [S] is given by

$$F_\Omega^S(z, X) = \sup \left\{ (L_u(z, X))^{1/2}; \quad u \in \mathcal{S}_z(\Omega) \right\}$$

where

$$L_u(z, X) = \sum_{j,k=1}^n \frac{\partial^2 u(z)}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k.$$

The Bergman metric is a Kähler metric and the Kobayashi and Sibony metrics are Finsler metrics. Note that while the Kobayashi metric is always upper semi-continuous ([R, Prop. 3 on p. 129], the Sibony metric is not, even for a domain of holomorphy (see [JP, Example 4.2.10]). Both the Kobayashi and Sibony metrics are identical to the Poincaré metric on the unit disk and satisfy the following length decreasing property: If $\Phi: \Omega_1 \rightarrow \Omega_2$ is a holomorphic map, then

$$F_{\Omega_1}(z, X) \geq F_{\Omega_2}(\Phi(z), \Phi_{*z}(X)).$$

Furthermore, $F_{\Omega}^S(z, X) \leq F_{\Omega}^K(z, X)$.

Catlin [C2] constructed bounded plurisubharmonic functions with large Hessians and used them to estimate the Bergman kernel and invariant metrics for smooth bounded pseudoconvex domains of finite type in \mathbb{C}^2 . The following theorem was proved by Catlin ([C2, Theorem 6.1 and pp. 461-462]), using Hörmander's L^2 -estimates for the $\bar{\partial}$ -equation. We first fix some notations. Denote by $D_j^{\alpha_j}$ any mixed partial derivative in z_j and \bar{z}_j of total order α_j . For $\alpha = (\alpha_1, \dots, \alpha_n)$, write $D^{\alpha}\phi = D_1^{\alpha_1} \dots D_n^{\alpha_n}\phi$.

Theorem 2.1 (Catlin). *Let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain and let $\hat{z} \in \Omega$. Let $\beta_1, \beta_2, \dots, \beta_n$ be given positive numbers. Assume that there exists a function $\phi \in C^3(\bar{\Omega})$ such that*

- (1) $|\phi(z)| \leq 1$, for $z \in \Omega$;
- (2) ϕ is plurisubharmonic in Ω ;
- (3) $P = \{z \in \mathbb{C}^n; |z_j - \hat{z}_j| < \beta_j, 1 \leq j \leq n\} \subset \Omega$;
- (4) for all $z \in P$ and $X \in \mathbb{C}^n$,

$$L_{\phi}(z, X) \gtrsim \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2};$$

- (5) for all α with $|\alpha| \leq 3$ and $z \in P$,

$$|D^{\alpha}\phi(z)| \lesssim \prod_{j=1}^n \beta_j^{-\alpha_j}.$$

Then there exists a positive constant C , independent of \hat{z} , such that

$$(2.1) \quad C^{-1} \prod_{j=1}^n \frac{1}{\beta_j^2} \leq K_{\Omega}(\hat{z}, \bar{\hat{z}}) \leq C \prod_{j=1}^n \frac{1}{\beta_j^2}$$

and

$$(2.2) \quad C^{-1} \left\{ \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2} \right\}^{1/2} \leq F_{\Omega}^B(\hat{z}, X) \leq C \left\{ \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2} \right\}^{1/2}.$$

The following proposition shows that same estimates also hold for the Kobayashi and Sibony metrics:

Proposition 2.2. *Let Ω be a domain in \mathbb{C}^n and let $\hat{z} \in \Omega$. Let $\beta_1, \beta_2, \dots, \beta_n$ be given positive numbers. Assume that there exists a function $\phi \in C^2(\Omega)$ such that*

- (1) $|\phi(z)| \leq 1$, for $z \in \Omega$;
- (2) ϕ is plurisubharmonic in Ω ;
- (3) $P = \{z \in \mathbb{C}^n; |z_j - \hat{z}_j| < \beta_j, 1 \leq j \leq n\} \subset \Omega$;
- (4) for all $z \in P$ and $X \in \mathbb{C}^n$,

$$L_{\phi}(z, X) \gtrsim \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2}.$$

Then there exists a positive constant C , independent of \hat{z} , such that

$$(2.3) \quad C^{-1} \left\{ \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2} \right\}^{1/2} \leq F_{\Omega}(\hat{z}, X) \leq C \left\{ \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2} \right\}^{1/2}$$

where F_Ω is either the Kobayashi or the Sibony metric.

Proof. It suffices to establish the upper bounded in (2.3) for the Kobayashi metric and the lower bound for the Sibony metric. The upper bound follows directly by comparing the Kobayashi metric on Ω with that on the polydisc P and using the length decreasing property. We now prove the lower bound for the Sibony metric, following [S].

Let $\chi(t) : \mathbb{R}^+ \rightarrow [0, 1]$ be a cut-off function such that $\chi \in C^2(\mathbb{R}^+)$, $\chi(t) = t$ for $t \in [0, 1/2]$, $\chi(t) = 1$ for $t \in [1, \infty)$, and $\chi''(t) \leq 0$ for $t \in \mathbb{R}^+$. It is easy to see that there exists a positive constant α so that

$$t(\log \chi)''(t) + \chi'(t) \geq -\alpha$$

for $t \in [1/2, 1]$.

Let

$$g(z) = \sum_{j=1}^n \frac{|z_j - \hat{z}_j|^2}{\beta_j^2}$$

and

$$u(z) = \chi(g(z)) \exp(M(\phi(z) - 1))$$

where M is a large constant to be chosen. Then $u(\hat{z}) = 0$ and $0 \leq u \leq 1$ in Ω . Evidently, $\log u$ is plurisubharmonic when $g(z) < 1/2$ or when $g > 1$. We now consider the case when $1/2 \leq g(z) \leq 1$. A simple computation yields that

$$\begin{aligned} L_{\log u}(z, X) &= L_{\log \chi(g)}(z, X) + ML_\phi(z, X) \\ &= (\log \chi)''(g(z)) |\langle \partial g(z), X \rangle|^2 + (\log \chi)'(g(z)) L_g(z, X) + ML_\phi(z, X). \end{aligned}$$

Since

$$|\langle \partial g(z), X \rangle|^2 \leq g(z) \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2}$$

and

$$L_g(z, X) = \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2},$$

it follows that

$$L_{\log u}(z, X) \geq (MC - \alpha) \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2},$$

where C is a positive constant. Choosing $M \geq \alpha/C$, we then obtain that $\log u$ is plurisubharmonic on Ω . From the definition of the Sibony metric, we then have

$$\begin{aligned} (F_\Omega^S(\hat{z}, X))^2 &\geq \exp(M(\phi(\hat{z}) - 1)) \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2} \\ &\geq e^{-2M} \sum_{j=1}^n \frac{|X_j|^2}{\beta_j^2}. \end{aligned}$$

We thus conclude the proof of (2.3). □

3. LEVI FOLIATIONS OF REAL HYPERSURFACES

We first recall well-known facts about local foliations of hypersurfaces whose Levi form has constant rank, following [Fr]. Let M be a smooth real hypersurface in \mathbb{C}^n . Let $z^0 \in M$ and let $r(z)$ be a local defining function of M on a neighborhood V of z^0 . The *Levi rank* of M at z^0 , denoted by $R(M, z^0)$, is the number of non-zero eigenvalues of the Levi form

$$L_r(z^0; X, Y) = \sum_{j,k=1}^n \frac{\partial^2 r(z^0)}{\partial z_j \partial \bar{z}_k} X_j \bar{Y}_k$$

for $X, Y \in T_{z^0}^{1,0}(M)$. The *Levi nullspace* \mathcal{N}_{z^0} of M at z^0 is given by

$$\mathcal{N}_{z^0} = \left\{ X \in T_{z^0}^{1,0}(M); L_r(z^0; X, Y) = 0 \text{ for all } Y \in T_{z^0}^{1,0}(M) \right\}.$$

Thus $R(b\Omega, z^0) = n - 1 - \dim_{\mathbb{C}} \mathcal{N}_{z^0}$. Note that both $R(b\Omega, z^0)$ and $\dim_{\mathbb{C}} \mathcal{N}_{z^0}$ are independent of the choices of the defining functions or local holomorphic coordinates. A *complex foliation* of (complex) codimension q of $M \cap V$ is a set \mathcal{F} of complex submanifolds of V such that there exists a smooth map $\sigma: V \rightarrow \mathbb{R}^{2q}$ of rank $2q$ on M satisfying:

- (1) $M \cap V = \{z \in V; \sigma_1(z) = 0\}$;
- (2) $\mathcal{F} = \cup \{M_c; c = (c_2, \dots, c_{2q}) \in \mathbb{R}^{2q-1}\}$, where

$$M_c = \{z \in M \cap V; \sigma_j(z) = c_j, 2 \leq j \leq 2q\}.$$

Each M_c is a *leaf* of the foliation. Note that \mathcal{F} is a complex foliation of codimension $n - l$ of $M \cap V$ defined by $\sigma = (\sigma_1, \dots, \sigma_{2(n-l)})$ if and only if for each $z \in M \cap V$, there exists a neighborhood $U \subset V$ of z and holomorphic functions f_{n-l+1}, \dots, f_n on U such that

- (1) the map $F = (\sigma_1, \dots, \sigma_{2(n-l)}, f_{n-l+1}, \dots, f_n)$ is a diffeomorphism from U onto an open subset $W = I \times I' \times W''$ of $\mathbb{R} \times \mathbb{R}^{2(n-l)-1} \times \mathbb{C}^l$;
- (2) $G = F^{-1}$ maps $\{0\} \times I' \times W''$ onto $M \cap U$;
- (3) $G_c(\cdot) = G(0, c, \cdot)$ is holomorphic on W'' for each $c \in I'$.

(See [Fr, Section 2].) The following theorem is well-known (cf. [Fr, Theorem 6.1]):

Theorem 3.1 (cf. [Fr]). *Let M be a real hypersurface in \mathbb{C}^n . Suppose M has constant constant Levi rank $n - l - 1$. Then for each $p \in M$, there exists a neighborhood V of p and a unique complex foliation of codimension $n - l$ of $M \cap V$ such that \mathcal{N}_z is the complex tangent space of leaves of the foliation.*

Let $\Omega \subset \subset \mathbb{C}^n$ be a domain and let $z^0 \in b\Omega$. Assume that $b\Omega$ is smooth in a neighborhood V of z^0 and that $R(b\Omega, z) = n - l - 1$ for all $z \in b\Omega \cap V$. Let

$$r(z) = \begin{cases} -\text{dist}(z, b\Omega), & z \in \Omega; \\ \text{dist}(z, b\Omega), & z \notin \Omega. \end{cases}$$

After a possible shrinking of V , we may assume that $r(z) \in C^\infty(V)$. For $z = (z_1, \dots, z_n)$, we write $\tilde{z} = (z_2, \dots, z_n)$, $z' = (z_2, \dots, z_{n-l})$, and $z'' = (z_{n-l+1}, \dots, z_n)$. The following proposition is a variation of Lemma 3.3.2 in [C1].

Proposition 3.2. *Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain and let $z^0 \in b\Omega$. Assume that $b\Omega$ is smooth in a neighborhood V of z^0 and that $R(b\Omega, z) = n - l - 1$ for all $z \in b\Omega \cap V$. Then there exists a neighborhood $U \subset \subset V$ of z^0 such that for each $p \in b\Omega \cap U$, there is a biholomorphism mapping $\zeta = \Phi_p(z)$ from U onto the unit ball $B(0, 1)$ that satisfies*

- (1) $\Phi_p(p) = 0$;
- (2) Φ_p depends smoothly on p ;
- (3) $\Phi_p(b\Omega \cap U)$ has a defining function of the form

$$(3.1) \quad \rho(\zeta) = \operatorname{Re} \zeta_1 + \sum_{j=2}^{n-l} \lambda_j |\zeta_j|^2 + O(|\zeta'|^2 \cdot |\zeta''| + |\zeta'|^3 + |\operatorname{Im} \zeta_1| \cdot |\zeta|)$$

near 0, where λ_j , $2 \leq j \leq n-l$, are positive constants depending smoothly on p .

Proof. By Theorem 3.1, there exists a complex foliation \mathcal{F} of codimension $n-l$ for $b\Omega \cap V$. Assume that the foliation \mathcal{F} is defined by $\sigma = (\sigma_1, \dots, \sigma_{2(n-l)})$. Then there exist a neighborhood $U \subset\subset V$ of z^0 and holomorphic functions f_{n-l+1}, \dots, f_n on U such that

- (1) the map $F = (\sigma_1, \dots, \sigma_{2(n-l)}, f_{n-l+1}, \dots, f_n)$ is a diffeomorphism from U onto an open subset $W = I \times I' \times W''$ of $\mathbb{R} \times \mathbb{R}^{2(n-l)-1} \times \mathbb{C}^l$;
- (2) $G = F^{-1}$ maps $\{0\} \times I' \times W''$ onto $b\Omega \cap U$;
- (3) $G_c(w'') = G(0, c, w'')$ is holomorphic on $w'' = (w_{n-l+1}, \dots, w_n) \in W''$ for each $c \in I'$.

For each $z \in V$, we write $c_z = (\sigma_2(z), \dots, \sigma_{2(n-l)}(z))$. Since the rank of the matrix

$$\left(\frac{\partial G_k}{\partial w_j}(0, c_{z^0}, w'') \right)_{\substack{1 \leq k \leq n \\ n-l+1 \leq j \leq n}}$$

is l , we may assume without loss of generality that

$$\det \left(\frac{\partial G_k}{\partial w_j}(0, c_{z^0}, w'') \right)_{\substack{n-l+1 \leq k \leq n \\ n-l+1 \leq j \leq n}} \neq 0.$$

After possible shrinking of U and W'' , we have

$$\det \left(\frac{\partial G_k}{\partial w_j}(0, c_p, w'') \right)_{\substack{n-l+1 \leq k \leq n \\ n-l+1 \leq j \leq n}} \neq 0$$

for all $p \in U \cap b\Omega$ and $w'' \in W''$. Therefore the map

$$w'' \mapsto (G_{n-l+1}(0, c_p, w''), \dots, G_n(0, c_p, w''))$$

is invertible on W'' . Let $z'' \mapsto (g_{n-l+1}(z''), \dots, g_n(z''))$ be its inverse. Let $\xi = \widehat{\Psi}_p(z)$ be defined by

$$\xi_j = z_j - G_j(0, c_p, g_{n-l+1}(z''), \dots, g_n(z'')), \quad 1 \leq j \leq n-l,$$

and

$$\xi_j = z_j, \quad n-l+1 \leq j \leq n.$$

Then $\widehat{\Psi}_p$ is a biholomorphic map from U into \mathbb{C}^n . After a rotation and a translation, we may assume that $\widehat{\Psi}_p(p) = 0$ and the positive $\operatorname{Re} \xi_1$ -axis is the outward normal direction at $\widehat{\Psi}_p(p)$ of $\widehat{\Psi}_p(b\Omega \cap U)$. By the implicit function theorem, $\widehat{\Psi}_p(b\Omega \cap U)$ is defined by

$$\hat{\rho}(\xi) = \operatorname{Re} \xi_1 + \hat{f}(\tilde{\xi}) + (\operatorname{Im} \xi_1) \hat{g}(\tilde{\xi}, \operatorname{Im} \xi_1)$$

for ξ in a neighborhood U_1 of 0, where

$$\hat{f}(\tilde{\xi}) = O(|\tilde{\xi}|^2); \quad \hat{g}(\tilde{\xi}, \operatorname{Im} \xi_1) = O(|\tilde{\xi}| + |\operatorname{Im} \xi_1|).$$

From property (2) of G above and the definition of $\widehat{\Psi}_p$, we have

$$\{\xi \in U; \xi_1 = \cdots = \xi_{n-l} = 0\} \subset \widehat{\Psi}_p(b\Omega \cap U).$$

Therefore, $\hat{f}(0, \xi'') \equiv 0$. Hence the complex Hessian of f at 0 has the form

$$(3.2) \quad \left(\frac{\partial \hat{f}(0)}{\partial \xi_j \partial \bar{\xi}_j} \right)_{2 \leq j, k \leq n} = \begin{pmatrix} B & A \\ \bar{A}^\tau & 0 \end{pmatrix}$$

where A is a $(n-l-1) \times l$ matrix and B is a $(n-l-1) \times (n-l-1)$ matrix.

It follows from the pseudoconvexity of Ω that the matrix in (3.2) is positive semi-definite. Therefore, B is also positive semi-definite and $A = 0$. After a unitary transformation in the (z_2, \dots, z_{n-l}) -variables, we may assume that

$$B = \begin{pmatrix} \lambda_2 & 0 & \cdots & 0 \\ 0 & \lambda_3 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_{n-l} \end{pmatrix}$$

where λ_j , $2 \leq j \leq n-l$, are positive constants. Thus the Taylor expansion of $\hat{f}(\tilde{\xi})$ at 0 has the form

$$\hat{f}(\tilde{\xi}) = \sum_{j=2}^{n-l} \lambda_j |\xi_j|^2 + 2 \operatorname{Re} \sum_{2 \leq j, k \leq n-l} \frac{\partial^2 \hat{f}(0)}{\partial \xi_j \partial \bar{\xi}_k} \xi_j \bar{\xi}_k + O(|\tilde{\xi}|^3)$$

for $\tilde{\xi}$ near 0.

Let $\zeta = \Psi_p(\xi)$ be defined by

$$\begin{aligned} \zeta_1 &= \xi_1 + 2 \sum_{2 \leq j, k \leq n-l} \frac{\partial^2 \hat{f}(0)}{\partial \xi_j \partial \bar{\xi}_k} \xi_j \bar{\xi}_k; \\ \tilde{\zeta} &= \tilde{\xi}. \end{aligned}$$

Let $\Phi_p = \Psi_p \circ \widehat{\Psi}_p$. Then $\Phi_p(b\Omega \cap U)$ has a defining function $\rho(\zeta) = \hat{\rho}(\Psi_p^{-1}(\zeta))$ of the form

$$(3.3) \quad \rho(\zeta) = \operatorname{Re} \zeta_1 + f(\tilde{\zeta}) + (\operatorname{Im} \zeta_1)g(\tilde{\zeta}, \operatorname{Im} \zeta_1)$$

near 0, where

$$(3.4) \quad f(\tilde{\zeta}) = \sum_{j=2}^{n-l} \lambda_j |\zeta_j|^2 + O(|\tilde{\zeta}|^3)$$

and

$$(3.5) \quad g(\tilde{\zeta}, \operatorname{Im} \zeta_1) = O(|\tilde{\zeta}| + |\operatorname{Im} \zeta_1|).$$

The Taylor expansion of $f(\tilde{\zeta})$ at $(0, \zeta'')$ has the form

$$(3.6) \quad \begin{aligned} f(\tilde{\zeta}) &= \sum_{2 \leq j, k \leq n-l} \frac{\partial^2 f}{\partial \zeta_j \partial \bar{\zeta}_k}(0, \zeta'') \zeta_j \bar{\zeta}_k \\ &+ 2 \operatorname{Re} \sum_{2 \leq j, k \leq n-l} \frac{\partial^2 f}{\partial \zeta_j \partial \zeta_k}(0, \zeta'') \zeta_j \zeta_k + O(|\zeta'|^3) \end{aligned}$$

for $\tilde{\zeta}$ near 0. Comparing (3.4) and (3.6), we have

$$\frac{\partial^2 f}{\partial \zeta_j \partial \bar{\zeta}_k}(0, \zeta'') = \lambda_j \varepsilon_{jk} + O(|\zeta''|)$$

and

$$\frac{\partial^2 f}{\partial \zeta_j \partial \zeta_k}(0, \zeta'') = O(|\zeta''|)$$

for $j, k \in \{2, \dots, n-l\}$, where $\varepsilon_{jk} = 0$ if $j \neq k$ and $\varepsilon_{jk} = 1$ if $j = k$. Thus it follows from (3.6) that

$$(3.7) \quad f(\tilde{\zeta}) = \sum_{j=2}^{n-l} \lambda_j |\zeta_j|^2 + O(|\zeta'|^2 \cdot |\zeta''| + |\zeta'|^3).$$

From (3.3), (3.5), and (3.7), we then know that $\Phi_p(b\Omega \cap U)$ has a defining function in the form of (3.1). \square

A boundary point p of Ω is called a *local weak peak point* if there exist a neighborhood U_p of p and a function f_p holomorphic on $\Omega \cap U_p$ and continuous on $\overline{\Omega} \cap U_p$ such that $f_p(p) = 1$, $|f_p(z)| < 1$ for $z \in \Omega \cap U_p$, and $|f_p(z)| \leq 1$ for $z \in \overline{\Omega} \cap U_p$. The function f_p is called a *local weak peak function* of Ω at p .

Corollary 3.3. *Assume the same hypotheses as in Proposition 3.2. Then each $p \in b\Omega \cap V$ is a local weak peak point of Ω .*

Proof. It follows from Proposition 3.2 that for any $p \in b\Omega \cap V$, there exist a neighborhood \widehat{U}_p of p and a biholomorphic mapping Φ_p from \widehat{U}_p onto $B(0, 1)$ such that

$$\Phi_p(\Omega \cap \widehat{U}_p) \cap B(0, \varepsilon_0) \subset \{\zeta \in B(0, \varepsilon_0); \operatorname{Re} \zeta_1 - |\operatorname{Im} \zeta_1| < 0\}$$

for a sufficiently small $\varepsilon_0 > 0$. Let

$$h(\zeta) = \exp(-(-\zeta_1)^{2/3})$$

where the cubic root takes the principle branch by deleting the negative $\operatorname{Re} \zeta_1$ -axis. Let $U_p = \Phi_p^{-1}(B(0, \varepsilon_0)) \cap \widehat{U}_p$. Then $f_p(z) = h(\Phi_p(z))$ is a local weak peak function at p defined on $\overline{\Omega} \cap U_p$. \square

Remark. Let Ω be a pseudoconvex domain with piecewise smooth boundary such that each piece has constant Levi rank. Then it follows from the localization and length decreasing properties of the Kobayashi metric ([R, p. 136]) and Corollary 3.3 that Ω is Kobayashi complete. This also follows from Theorem 1.1 (to be proved in Section 5).

4. CONSTRUCTION OF PLURISUBHARMONIC FUNCTIONS WITH LARGE HESSIANS

We now turn to the construction of bounded plurisubharmonic functions with large Hessians near a piece of boundary that has constant Levi rank on a pseudoconvex domain Ω . For $\delta, a > 0$, and $X \in \mathbb{C}^n$, let

$$P_{\delta,a} = \{\zeta \in \mathbb{C}^n; |\zeta_1| < a\delta, |\zeta_j| < a\delta^{\frac{1}{2}}, 2 \leq j \leq n-l, \\ |\zeta_j| < a, n-l+1 \leq j \leq n\}$$

and let

$$\omega(X, \delta) = \frac{|X_1|^2}{\delta^2} + \sum_{j=2}^{n-l} \frac{|X_j|^2}{\delta} + \sum_{j=n-l+1}^n |X_j|^2.$$

We will follow the notations in Section 3. For $p \in b\Omega \cap U$, let $\widetilde{\Omega}_p = \Phi_p(\Omega \cap U)$. The following construction of plurisubharmonic functions with large Hessians plays a key role in this paper (compare [C2, Prop. 2.1]; also [S, Prop. 7]).

Theorem 4.1. *Assume the hypothesis of Proposition 3.2. Let $W \subset\subset U$ be a neighborhood of z^0 . Then for any $p \in b\Omega \cap W$ and any sufficiently small δ , there exists a function $g_{p,\delta} \in C^\infty(\tilde{\Omega}_p)$ and constants a, b, C and C_α , independent of p and δ , such that*

- (1) $|g_{p,\delta}(z)| \leq 1, z \in \tilde{\Omega}_p;$
- (2) $g_{p,\delta}$ is plurisubharmonic on $\tilde{\Omega}_p;$
- (3) for $\zeta \in P_{\delta,ab} \cap \tilde{\Omega}_p$ and $Y \in \mathbb{C}^n$,

$$L_{g_{p,\delta}}(\zeta, Y) \geq \frac{1}{C} \left(\frac{|\langle \partial\rho(\zeta), Y \rangle|^2}{\delta^2} + \sum_{j=2}^{n-l} \frac{|X_j|^2}{\delta} + \sum_{j=n-l+1}^n |X_j|^2 \right);$$

- (4) for $\zeta \in P_{\delta,ab} \cap \tilde{\Omega}_p$,

$$|D^\alpha g_{p,\delta}(\zeta)| \leq C_\alpha \delta^{-(\alpha_1 + \frac{1}{2} \sum_{j=2}^{n-l} \alpha_j)}$$

where $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$.

Proof. Let $\chi_1(t) \in C^\infty(\mathbb{R})$ be a decreasing function with $\chi_1(t) = 1$ for $t < \frac{1}{2}$ and $\chi_1(t) = 0$ for $t > 1$. Let $\phi_\delta(\zeta)$ be defined by

$$\phi_\delta(\zeta) = \chi_1 \left(\frac{1}{a^2} \omega(\zeta, \delta) \right)$$

and let $G_{p,\delta}$ be defined by

$$G_{p,\delta}(\zeta) = \phi_\delta(\zeta) e^{\frac{M}{\delta} \rho(\zeta)}$$

where $\rho(\zeta)$ is the defining function obtained from Proposition 3.2 and M is a large constant to be chosen.

For $Y = (Y_1, \dots, Y_n) \in \mathbb{C}^n$, let

$$Y_1^* = Y_1 - \sum_{j=2}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta) \left(\frac{\partial \rho}{\partial \zeta_1}(\zeta) \right)^{-1} Y_j;$$

$$\tilde{Y}^* = \tilde{Y},$$

and $Y^* = (Y_1^*, \tilde{Y}^*)$. It follows from a direct calculation that

$$(4.1) \quad \begin{aligned} L_{G_{p,\delta}}(\zeta, Y^*) &= e^{\frac{M}{\delta} \rho(\zeta)} [L_{\phi_\delta}(\zeta, Y^*) + 2 \frac{M}{\delta} \operatorname{Re} \sum_{j,k=1}^n \frac{\partial \phi_\delta}{\partial \zeta_j}(\zeta) Y_j^* \overline{\frac{\partial \rho}{\partial \zeta_k}(\zeta) Y_k^*} \\ &\quad + \frac{M}{\delta} \phi_\delta L_\rho(\zeta, Y^*) + \left(\frac{M}{\delta} \right)^2 \phi_\delta(\zeta) |\langle \partial\rho(\zeta), Y^* \rangle|^2]. \end{aligned}$$

It follows from (3.1) that when a is sufficiently small,

$$\left| \frac{\partial \rho}{\partial \zeta_1}(\zeta) \right| \approx 1; \quad \left| \frac{\partial \rho}{\partial \zeta_j}(\zeta) \right| \lesssim \delta^{\frac{1}{2}}, \quad 2 \leq j \leq n-l; \quad \left| \frac{\partial \rho}{\partial \zeta_j}(\zeta) \right| \lesssim \delta, \quad n-l+1 \leq j \leq n$$

for $\zeta \in P_{\delta,a}$. Therefore

$$(4.2) \quad |\langle \partial\rho(\zeta), Y^* \rangle|^2 = \left| \frac{\partial \rho}{\partial \zeta_1}(\zeta) Y_1^* \right|^2 \approx |Y_1|^2$$

for $\zeta \in P_{\delta,a}$. Furthermore,

$$(4.3) \quad L_\rho(\zeta, Y^*) \gtrsim |Y'|^2 - C \left(\delta |Y''|^2 + \frac{|Y_1|^2}{\delta} \right)$$

for $\zeta \in P_{\delta,a}$.

For the second term on the right hand side of (4.1), we have

$$(4.4) \quad \begin{aligned} \left| \frac{M}{\delta} \operatorname{Re} \sum_{j,k=1}^n \frac{\partial \phi_\delta}{\partial \zeta_j}(\zeta) Y_j^* \overline{\frac{\partial \rho}{\partial \zeta_k}(\zeta) Y_k^*} \right| &= \left| \frac{M}{\delta} \operatorname{Re} \left(\sum_{j=1}^n \frac{\partial \phi_\delta}{\partial \zeta_j}(\zeta) Y_j^* \right) \cdot \left(\frac{\partial \rho}{\partial \zeta_1}(\zeta) Y_1 \right) \right| \\ &\lesssim M^{\frac{3}{2}} \frac{|Y_1|^2}{\delta^2} + M^{\frac{1}{2}} \left(\frac{|Y'|^2}{\delta} + |Y''|^2 \right) \end{aligned}$$

for $\zeta \in P_{\delta,a}$. Combining (4.1)–(4.4), we obtain

$$(4.5) \quad L_{G_{p,\delta}}(\zeta, Y^*) \gtrsim \frac{|Y_1|^2}{\delta^2} + \frac{|Y'|^2}{\delta} - C |Y''|^2$$

if $\phi_\delta(\zeta) > \frac{1}{4}$ and M is sufficiently large.

Let $\chi_2(t) \in C^\infty(\mathbb{R})$ be a convex increasing function such that $\chi_2(t) = 0$ for $t < \frac{1}{2}$ and $\chi_2(t) > 0$, $\chi_2'(t) > 0$ for $t > \frac{1}{2}$. Let $\hat{g}_{p,\delta}(\zeta)$ be defined by

$$\hat{g}_{p,\delta}(\zeta) = \chi_2(G_{p,\delta}(\zeta)).$$

If $\zeta \in \tilde{\Omega}_p$ and $G_{p,\delta}(\zeta) > \frac{1}{4}$, then $\phi_\delta(\zeta) > \frac{1}{4}$. Therefore it follows from (4.5) that

$$(4.6) \quad \begin{aligned} L_{\hat{g}_{p,\delta}}(\zeta, Y^*) &= \chi_2''(G_{p,\delta}(\zeta)) |\langle \partial G_{p,\delta}(\zeta), Y^* \rangle|^2 + \chi_2'(G_{p,\delta}(\zeta)) L_{G_{p,\delta}}(\zeta, Y^*) \\ &\gtrsim \chi_2'(G_{p,\delta}) \left(\frac{|Y_1|^2}{\delta^2} + \frac{|Y'|^2}{\delta} - C |Y''|^2 \right). \end{aligned}$$

If $\zeta \in P_{\delta,ab} \cap \tilde{\Omega}_p$ and b is sufficiently small, then $\phi_p(\zeta) = 1$ and $\rho(\zeta) > -\frac{\delta}{2M}$. Thus,

$$G_{p,\delta}(\zeta) = e^{\frac{M}{\delta}\rho(\zeta)} > e^{-\frac{1}{2}} > \frac{1}{2}.$$

Therefore it follows from (4.6) that

$$L_{\hat{g}_{p,\delta}}(\zeta, Y^*) \gtrsim \chi_2'(e^{-\frac{1}{2}}) \left(\frac{|Y_1|^2}{\delta^2} + \frac{|Y'|^2}{\delta} - C |Y''|^2 \right).$$

Let $g_{p,\delta}(\zeta) = \frac{1}{C_1}(\hat{g}_{p,\delta}(\zeta) + C_2|\zeta|^2)$. Then when C_1 and C_2 are sufficiently large, $g_{p,\delta}(\zeta)$ satisfies properties (1)–(4) of Theorem 4.1. \square

5. PROOF OF THE MAIN THEOREM

We prove Theorem 1.1 in this section. Let $p \in b\Omega \cap W$. Following the notations of Theorem 4.1, we write $\tilde{\Omega}_p = \Phi_p(\Omega \cap U)$. Denote

$$Q_{\delta,c} = \left\{ \zeta \in \mathbb{C}^n; \begin{aligned} &|\zeta_1 - c\delta| < c^2\delta, \quad |\zeta_j| < c^2\delta^{\frac{1}{2}}, \quad 2 \leq j \leq n-l, \\ &|\zeta_j| < c^2, \quad n-l+1 \leq j \leq n \end{aligned} \right\}.$$

It follows from (3.1) that, when c and δ are sufficiently small,

$$(5.1) \quad Q_{\delta,c} \subset \tilde{\Omega}_p \cap P_{\delta,ab}.$$

It is easy to see that

$$|Y_1|^2 \lesssim |\langle \partial \rho, Y \rangle|^2 + c \left(\delta^{\frac{1}{2}} |Y'|^2 + \delta |Y''|^2 \right)$$

on $Q_{\delta, c}$. Therefore, by choosing c sufficiently small, we have

$$L_{g_{p, \delta}}(\zeta, Y) \gtrsim \omega(Y, \delta).$$

Applying Theorems 2.1 and 4.1, we have

$$(5.2) \quad K_{\tilde{\Omega}_p}(\zeta_\delta, \zeta_\delta) \approx \frac{1}{\delta^{n-l+1}}$$

where $\zeta_\delta = (-c\delta, 0)$.

Let $p_\delta = \Phi_p^{-1}(\zeta_\delta)$. Since $|J\Phi_p(p_\delta)| \approx 1$, it follows from the localization property of the Bergman kernel (see [O1]) and (5.2) that

$$\begin{aligned} K_\Omega(p_\delta, p_\delta) &\approx K_{\Omega \cap U}(p_\delta, p_\delta) \\ &= K_{\tilde{\Omega}_p}(\zeta_\delta, \zeta_\delta) |J\Phi_p(p_\delta)|^2 \\ &\approx \frac{1}{\delta^{n-l+1}}. \end{aligned}$$

when δ is sufficiently small. By letting p vary on $b\Omega \cap W$ for a small neighborhood $W \subset \subset U$ of z^0 and letting δ vary in $(0, \varepsilon_0)$ for a sufficiently small $\varepsilon_0 > 0$, we obtain the estimates for the Bergman kernel in Theorem 1.1.

Estimates for the Bergman and Kobayashi metrics are proved similarly as above by applying Theorem 4.1, Theorem 2.1, and Proposition 2.2. We provide only the detail for the Kobayashi metric. Let $Y = \Phi_{p*}(X)$. By the localization property of the Kobayashi metric (see [R, Lemma 2]), we have

$$(5.3) \quad \begin{aligned} F_\Omega^K(p_\delta, X) &\approx F_{\Omega \cap U}^K(p_\delta, X) \\ &= F_{\tilde{\Omega}_p}^K(\zeta_\delta, Y). \end{aligned}$$

It follows from (5.1), Proposition 2.2, and Theorem 4.1 that

$$(5.4) \quad \left(F_{\tilde{\Omega}_p}^K(\zeta_\delta, Y) \right)^2 \approx \omega(Y, \delta)$$

for sufficiently small δ .

From the definitions of Φ_p and Y , it is easy to see that

$$(5.5) \quad \omega(Y, \delta) - C|X|^2 \lesssim \frac{|L_\delta(p_\delta, X)|}{|\delta(p_\delta)|} + \frac{|\langle \partial \delta(p_\delta), X \rangle|^2}{|r(p_\delta)|^2} \lesssim \omega(Y, \delta) + C|X|^2$$

for a sufficiently large C (cf. [F2]). This concludes the proof of Theorem 1.1.

Remark. Theorem 1.1 gives the following characterization of Levi-flatness using the Bergman kernel and invariant metrics¹: Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain and let $\delta(z)$ be the Euclidean distance to $b\Omega$. Assume that $b\Omega$ is smooth in a neighborhood V of $z^0 \in b\Omega$. Then $b\Omega$ is Levi-flat near z^0 if and only if there exists a neighborhood $W \subset \subset V$ of z^0 and a positive constant C such that either (5.6) or (5.7) below holds:

$$(5.6) \quad \frac{1}{C} |\delta(z)|^{-2} \leq K_\Omega(z, z) \leq C |\delta(z)|^{-2};$$

¹For the Bergman metric, this was an open problem in a recent paper of Ohsawa [O2, Question 3].

$$(5.7) \quad \frac{1}{C} \left(\frac{|\langle \partial \delta(z), X \rangle|^2}{|\delta(z)|^2} + |X|^2 \right) \leq (F_\Omega(z, X))^2 \leq C \left(\frac{|\langle \partial \delta(z), X \rangle|^2}{|\delta(z)|^2} + |X|^2 \right)$$

for all $z \in W \cap \Omega$ and $X \in T_z^{1,0}(\Omega)$, where $F_\Omega(z, X)$ is either the Bergman or the Kobayashi metric.

The sufficiency is a special case of Theorem 1.1. To see the necessity, one observes that if $b\Omega$ is not Levi-flat near z^0 , then by a simple continuity and inductive argument on the Levi rank, z^0 is an accumulation point of boundary points z^k such that $b\Omega$ has constant Levi rank ≥ 1 near each z^k . We then arrive at a contradiction to Theorem 1.1.

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